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2003 J. Phys. A: Math. Gen. 36 10505

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Classical dynamical systems from q -algebras: ‘cluster’ variables and explicit solutions

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Received 7 July 2003

Published 7 October 2003

Online at stacks.iop.org/JPhysA/36/10505

Abstract

A general procedure to get the explicit solution of the equations of motion for N -body classical Hamiltonian systems equipped with coalgebra symmetry is introduced by defining a set of appropriate collective variables which are based on the iterations of the coproduct map on the generators of the algebra. In this way several examples of N -body dynamical systems obtained from q -Poisson algebras are explicitly solved: the q -deformed version of the $sl(2)$ Calogero–Gaudin (q CG) system, a q -Poincaré Gaudin system and a system of Ruijsenaars type arising from the same (non-coboundary) q -deformation of the $(1 + 1)$ Poincaré algebra. While the complete integrability of all these systems was already well known, being in fact encoded in their construction, no explicit solution was available until now. In particular, it turns out that there exists an open subset of the whole phase space where the orbits of the q CG system are periodic with the same period. Also, a unified interpretation of all these systems as different Poisson–Lie dynamics on the same three-dimensional solvable Lie group is given.

PACS numbers: 02.20.Uw, 02.30.Ik, 45.50.Jf

1. Introduction

The coalgebra approach [1, 2], and more recently its comodule algebra generalization [5] have been proved to be quite effective methods for constructing both classical and quantum Hamiltonian systems with a large family of commuting integrals of the motion. In some important cases (e.g. for rank 1 Lie algebras and their q -analogues) the coalgebra method leads to the proof of the complete integrability and even of the ‘super-integrability’ [6] of such kind of systems. Moreover, in the quantum mechanical case both non-deformed and q -deformed systems can be handled and explicitly solved on the same footing [7, 8].

Surprisingly enough, such an explicit solution looked difficult to obtain at the classical (Poisson) level, inasmuch as some traditional ingredients of classical integrability (such as the Lax representation) were missing for all these systems. This issue has been recently addressed in [9], where some steps towards the solution of the q -deformed version of the $sl(2)$ Calogero–Gaudin (qCG) system were accomplished.

In this paper we go ahead along the lines introduced in [9] and we present a general approach to the explicit solution of N -body classical Hamiltonian systems with coalgebra symmetry. This is achieved in the next section by using the coalgebra structure in order to define a set of collective variables (the so-called ‘cluster’ variables) whose equations of motion can be first separated by making use of the integrals for the system and then solved. Essentially, the coalgebra symmetry implies a kind of self-similarity of the dynamical equations for these cluster variables, and the $N = 2$ case turns out to be the essential cornerstone in order to solve the full N -body dynamics.

This general procedure is applied to several coalgebra systems in section 3. There we exhibit the complete solution of the classical equations of motion for the qCG system, which has the N -body Casimir function as the Hamiltonian, unveiling the deep connection between non-deformed and q -deformed systems. Afterwards, two more systems defined on a non-coboundary q -deformation of the $(1 + 1)$ Poisson Poincaré algebra are explicitly solved: the first is a q -Poincaré analogue of the CG system and the second is the Ruijsennars–Schneider-like system introduced in [2].

Finally, section 4 presents a unified interpretation of all the previous systems as different cases of dual Poisson–Lie (PL) dynamics on N copies of the three-dimensional solvable Lie group G_z generated by two translations and one dilation. From this point of view the coproduct map (which is formally the same for all the systems here considered) is just the Lie group multiplication of G_z , and the m th cluster variables are essentially the entries of the matrix representation of the product of m different G_z group elements. Therefore, the qCG system and the two q -Poincaré systems here solved are just two particular cases among the set of all dynamical systems defined by the Poisson-Lie structures on G_z . All such systems are, by construction, compatible with the G_z group multiplication but exhibit quite different dynamical features, as will be shown through the examples here introduced.

2. Coalgebras and cluster dynamics

We recall from [2] that the following general result can be proved:

Let $(A, \Delta^{(2)})$ be a (Poisson) coalgebra with generators X_k ($k = 1, \dots, l$) and with Casimirs C_i ($i = 1, \dots, r$). Then, the m th coproducts of the Casimirs (Poisson) commute among themselves and with the N th coproducts of the generators of A :

$$\begin{aligned} \{\Delta^{(m)}(C_i), \Delta^{(n)}(C_j)\} &= 0 & m, n = 1, \dots, N & \quad \forall i, j \\ \{\Delta^{(m)}(C_i), \Delta^{(N)}(X_k)\} &= 0 & m = 1, \dots, N & \quad \forall k \quad \forall i. \end{aligned} \quad (1)$$

Here the symbol $\{\cdot, \cdot\}$ simultaneously denotes both the commutator (for quantum systems where the algebra of observables A is non-commutative) and its Poisson analogue for classical mechanical systems. Note that with this notation the $\Delta^{(1)}$ coproduct is just the identity map. Hence, for rank $r = 1$ coalgebras (and their Poisson analogues), we can consider as a Hamiltonian any (smooth) function H of the three generators of the algebra:

$$H = H(X_1, X_2, X_3) \quad (2)$$

and the complete integrability is thus established for the Hamiltonian $H^{(N)}$ defined as the N th coproduct of H :

$$H^{(N)} = H(\Delta^{(N)}(X_1), \Delta^{(N)}(X_2), \Delta^{(N)}(X_3)). \tag{3}$$

Note that the Casimir C is just a particular function of the generators, and we could take $H \equiv C$ as a particular type of system (the so-called Gaudin-type Hamiltonians).

As extensively explained in [2], the proof of the above-mentioned statement relies on the fact that the coproduct $\Delta^{(2)}$ is a coassociative homomorphism: this means that there exist two equivalent ways to define the third-order coproduct map, namely

$$\Delta^{(3)} := (id \otimes \Delta^{(2)}) \circ \Delta^{(2)} = (\Delta^{(2)} \otimes id) \circ \Delta^{(2)} \tag{4}$$

and, in general, $(N - 1)$ equivalent definitions of the N th-order coproduct are possible:

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \circ \Delta^{(2)} \quad m = 1, \dots, N - 1. \tag{5}$$

This property holds for the Poisson case as well as in the quantum context, both in the non-deformed and in the q -deformed cases, and it underlies the superintegrability properties of the dynamical systems with coalgebra symmetry [6, 10].

In this paper we address the problem of the explicit solution of the classical dynamics of such N -body Hamiltonian systems. For this purpose, the choice of an appropriate set of dynamical variables will be essential. In this respect, we can rewrite expression (5) by using Sweedler’s notation [11] in which the coproduct $\Delta^{(2)}$ of an arbitrary element Y of the algebra A (with arbitrary rank r) is given as the linear combination

$$\Delta^{(2)}(Y) = \sum_{\alpha} Y_{1\alpha} \otimes Y_{2\alpha} \tag{6}$$

where $Y_{1\alpha}$ and $Y_{2\alpha}$ will be certain functions that live on two different copies of the algebra A :

$$Y_{1\alpha} = X_{1\alpha}(X_1, \dots, X_l) \quad Y_{2\alpha} = X_{2\alpha}(X_1, \dots, X_l). \tag{7}$$

From (5) we immediately get that, in this notation, the N th coproduct of Y reads

$$\Delta^{(N)}(Y) := \sum_{\alpha} \Delta^{(m)}(Y_{1\alpha}) \otimes \Delta^{(N-m)}(Y_{2\alpha}) \quad m = 1, \dots, N - 1 \tag{8}$$

and since any $\Delta^{(p)}$ map is an algebra homomorphism, we can write

$$\Delta^{(N)}(Y) := \sum_{\alpha} Y_{1\alpha}(\Delta^{(m)}(X_1), \dots, \Delta^{(m)}(X_l)) \otimes Y_{2\alpha}(\Delta^{(N-m)}(X_1), \dots, \Delta^{(N-m)}(X_l)) \tag{9}$$

where $m = 1, \dots, N - 1$.

Expression (9) leads in a natural way to the definition of the following set of $2l(N - 1)$ collective ‘cluster’ variables:

$$X_k^{(m)} := \Delta^{(m)}(X_k) \quad X_k^{(N-m)} := \Delta^{(N-m)}(X_k) \tag{10}$$

where $k = 1, \dots, l$ and $m = 1, \dots, N - 1$. With them (9) is rewritten as

$$\Delta^{(N)}(Y) := \sum_{\alpha} Y_{1\alpha}(X_1^{(m)}, \dots, X_l^{(m)}) \otimes Y_{2\alpha}(X_1^{(N-m)}, \dots, X_l^{(N-m)}) \tag{11}$$

for any fixed value of m . We stress that the variables $X_k^{(N-m)}$ live on the $(N - m)$ tensor copies of A which are located starting from the right of the full chain of N copies of A :

$$\overbrace{A \otimes A \otimes \dots \otimes A}^m \otimes \overbrace{A \otimes \dots \otimes A}^{N-m} \otimes A.$$

Therefore,

$$\{X_k^{(m)}, X_p^{(N-m)}\} = 0 \quad \forall k, p. \tag{12}$$

Moreover, since the coproduct is always a Poisson algebra homomorphism, the set of variables $\{X_1^{(m)}, \dots, X_l^{(m)}\}$ reproduces again the Poisson algebra A and the same is true for the complementary set $\{X_1^{(N-m)}, \dots, X_l^{(N-m)}\}$. In this notation the m th coproducts of Casimirs, which are constants of motion for $H^{(N)}$, are written as

$$C^{(m)} = \Delta^{(m)}(C) = C(X_1^{(m)}, \dots, X_l^{(m)}). \quad (13)$$

By construction, there also exists a complementary set of integrals in involution defined by

$$C^{(N-m)} = \Delta^{(N-m)}(C) = C(X_1^{(N-m)}, \dots, X_l^{(N-m)}). \quad (14)$$

These additional integrals explicitly show the intrinsic superintegrability properties of coalgebra symmetric systems [6, 10].

Now, the Hamiltonian $H^{(N)}$ is just a particular case of (9), and we can write it in $(N-1)$ equivalent ways labelled by m :

$$H^{(N)} = H(X_1^{(m)}, \dots, X_l^{(m)}; X_1^{(N-m)}, \dots, X_l^{(N-m)}) \quad m = 1, \dots, N-1. \quad (15)$$

From this perspective it is clear that the dynamics of the system will be explicitly known if we are able to solve the evolution equations for the cluster variables $X_k^{(m)}$, which are $(N-1)$ sets of l nonlinear coupled first-order ODEs

$$\dot{X}_k^{(m)} = \{X_k^{(m)}, H^{(N)}\} = f_k(X_1^{(m)}, \dots, X_l^{(m)}; X_1^{(N-m)}, \dots, X_l^{(N-m)}) \quad (16)$$

where, in turn, the complementary cluster variables $X_k^{(N-m)}$ fulfil the equations

$$\dot{X}_k^{(N-m)} = \{X_k^{(N-m)}, H^{(N)}\} = g_k(X_1^{(m)}, \dots, X_l^{(m)}; X_1^{(N-m)}, \dots, X_l^{(N-m)}). \quad (17)$$

As a final step, we will also have to solve the l coupled equations that provide the dynamics of the N th coproducts of the generators:

$$\dot{X}_k^{(N)} = \{X_k^{(N)}, H^{(N)}\} = h_k(X_1^{(N)}, \dots, X_l^{(N)}). \quad (18)$$

Note that the dynamics of the latter N -body collective cluster variables turns out to be different from any other lower m th dimensional $m < N$ set of cluster variables.

Despite the apparent complexity of the problem, we stress that all the $(N-1)$ sets of equations (16) are formally identical for any $m = 1, \dots, N-1$, since the functions f_k (and g_k) do not depend on m (this follows from the fact that the coproduct is a homomorphism). As a consequence, the dynamics of the collective cluster variables is the same whatever the number m of degrees of freedom of the cluster. This ‘self-similar’ dynamical behaviour turns out to be an absolutely general feature of Poisson coalgebras and holds for both non-deformed and deformed coalgebra symmetric systems. In the latter case the cluster structure of the dynamics is preserved by the q -deformation, but the (always long-range) interactions coming from the q -deformation implies a more involved dynamics.

As we shall see in the following sections, the strategy followed in order to solve the system (16) will be to eliminate the complementary variables $X_k^{(N-m)}$ by making use of the constants of motion given by the coalgebra symmetry (therefore, without working out the explicit solution for (17)). In particular, a distinguished set of dynamical systems appears when the Hamiltonian function $H^{(N)}$ is just the N th Casimir $C^{(N)}$ (the so-called Gaudin Hamiltonians). In this particular case, since the Poisson brackets (18) vanish, the N th coproducts of the generators $X_k^{(N)}$ give a set of l constants of motion that we shall call δ_k . Moreover, the expressions of the N th coproducts $X_k^{(N)}$ obtained by using (9) give rise to l different algebraic equations that, for a given m , involve both the $X_k^{(m)}$ and the $X_k^{(N-m)}$ variables. When the latter are eliminated from them by using such constants, the system (16) is transformed into

$$\dot{X}_k^{(m)} = \{X_k^{(m)}, H^{(N)}\} = \tilde{f}_k(X_1^{(m)}, \dots, X_l^{(m)}; \delta_1, \dots, \delta_l) \quad (19)$$

which represents m copies of the same one-body dynamics in an ‘external’ field (parametrized by $\delta_1, \dots, \delta_l$) and thus can be simultaneously solved for any value of m . In case we deal with more general Hamiltonians different from the N th Casimir, the lower dimensional ‘cluster Casimirs’ $C^{(m)}$ and $C^{(N-m)}$ will provide alternative integrals that can be used in order to eliminate the $X_k^{(N-m)}$ variables from equations (16).

Finally, we recall that in all the previous constructions of coalgebra systems (see for instance [4]), the Hamiltonians have been written in terms of canonical coordinates coming from specific symplectic realizations of the algebra A :

$$X_k = X_k(q_1, \dots, q_n; p_1, \dots, p_n; c_1, \dots, c_r) \quad k = 1, \dots, l \quad (20)$$

where c_i are the values of the Casimirs of A that define the symplectic manifold on which the functions (20) close the algebra A . The number n of pairs of canonical variables needed in order to obtain such a realization depends both on the number of generators l of A and of its rank r . For $r = 1$, it turns out that $n = 1$ and then the cluster variable $X_k^{(m)}$ would be a function of just m canonical pairs. However, for higher rank algebras $X_k^{(m)}$ will depend on the number nm of canonical coordinates and momenta, and equations (16) will give us the ‘global’ dynamics of the cluster variables encompassing simultaneously nm degrees of freedom, whose individual time evolution can only be obtained by inverting (if possible) algebraic equations (20).

3. Explicit solutions of coalgebra systems

3.1. The non-deformed CG system

As a first ‘toy model’ with $sl(2)$ coalgebra symmetry, we will solve the dynamics of the Calogero–Gaudin system [12–15] by using the approach of the previous section.

Let us consider the $sl(2)$ Poisson coalgebra

$$\{X_3, X_{\pm}\} = \pm 2X_{\pm} \quad \{X_+, X_-\} = X_3 \quad (21)$$

equipped with the primitive coproduct

$$\Delta(X_i) = X_i \otimes id + id \otimes X_i \quad i = 3, \pm \quad (22)$$

and with Casimir function:

$$C = \frac{1}{4}X_3^2 + X_+X_- \quad (23)$$

We consider the N th coproducts of the generators

$$\begin{aligned} X_i^{(N)} := \Delta^{(N)}(X_i) &= X_i \otimes \overbrace{id \otimes id \otimes \dots \otimes id}^{(N-1)} \\ &+ id \otimes X_i \otimes \overbrace{id \otimes \dots \otimes id}^{(N-2)} + \dots + \overbrace{id \otimes id \otimes \dots \otimes id}^{(N-1)} \otimes X_i \end{aligned} \quad (24)$$

and the non-deformed CG system is defined by the Hamiltonian

$$H^{(N)} := \Delta^{(N)}(C) = C^{(N)} = \frac{1}{4}(X_3^{(N)})^2 + X_+^{(N)}X_-^{(N)}. \quad (25)$$

The integrals of motion for $H^{(N)}$ are, by construction [4], the N th coproducts of the three generators $X_i^{(N)}$ and the m th coproducts of the Casimir

$$C^{(m)} = \frac{1}{4}(X_3^{(m)})^2 + X_+^{(m)}X_-^{(m)} \quad m = 1, \dots, (N-1) \quad (26)$$

together with the complementary integrals

$$C^{(N-m)} = \frac{1}{4}(X_3^{(N-m)})^2 + X_+^{(N-m)}X_-^{(N-m)} \quad m = 1, \dots, (N-1) \quad (27)$$

leading to the well-known superintegrability of the CG system.

An essential property of the primitive coproduct (24) is that, by using (9), it can be expressed in $(N - 1)$ equivalent forms in terms of the cluster variables (10):

$$X_i^{(N)} = X_i^{(m)} + X_i^{(N-m)} \quad m = 1, \dots, (N - 1). \tag{28}$$

The time evolution of such cluster variables is given by (16) and reads

$$\begin{aligned} \dot{X}_3^{(m)} &= 2(X_+^{(m)} X_-^{(N-m)} - X_-^{(m)} X_+^{(N-m)}) \\ \dot{X}_+^{(m)} &= X_3^{(m)} X_+^{(N-m)} - X_+^{(m)} X_3^{(N-m)} \\ \dot{X}_-^{(m)} &= -X_3^{(m)} X_-^{(N-m)} + X_-^{(m)} X_3^{(N-m)}. \end{aligned} \tag{29}$$

Now by using the integrals of motion given by the N th coproducts of the generators

$$X_i^{(N)} = X_i^{(m)} + X_i^{(N-m)} \equiv \delta_i \quad i = 3, \pm \tag{30}$$

the variables $X_i^{(N-m)}$ can be eliminated and equations (29) are transformed into the linear system:

$$\begin{aligned} \dot{X}_3^{(m)} &= 2(X_+^{(m)} \delta_- - X_-^{(m)} \delta_+) \\ \dot{X}_+^{(m)} &= X_3^{(m)} \delta_+ - X_+^{(m)} \delta_3 \\ \dot{X}_-^{(m)} &= -X_3^{(m)} \delta_- + X_-^{(m)} \delta_3. \end{aligned} \tag{31}$$

The motion for this system can be either hyperbolic or periodic, since the eigenvalues of the previous system are related to the value of the N th coproduct of the Casimir in the form

$$\omega = \pm 2\sqrt{\frac{1}{4}\delta_3^2 + \delta_+ \delta_-} = \pm 2\sqrt{C^{(N)}}. \tag{32}$$

Note that this result is valid for any ‘size’ m of the cluster variables. Finally, we point out that equations (31) (and therefore the integration constants) are not independent, since they are constrained by the m -cluster Casimir (26).

3.2. Dynamics of the q -Calogero–Gaudin system

The ‘standard’ q -deformation of $sl(2)$ is given by the Poisson brackets

$$\{X_3, X_\pm\} = \pm 2X_\pm \quad \{X_+, X_-\} = \frac{\sinh(zX_3)}{z} \tag{33}$$

equipped with the q -deformed coproduct ($q = e^z$)

$$\begin{aligned} \Delta(X_3) &= X_3 \otimes id + id \otimes X_3 \\ \Delta(X_\pm) &= e^{-\frac{z}{2}X_3} \otimes X_\pm + X_\pm \otimes e^{\frac{z}{2}X_3}. \end{aligned} \tag{34}$$

The q -deformed Casimir function reads

$$C_z = \frac{1}{4} \left\{ \frac{\sinh\left(\frac{z}{2}X_3\right)}{z/2} \right\}^2 + X_+ X_- \tag{35}$$

The N th coproduct $\Delta^{(N)}(X_3)$ is just of type (24). For the X_\pm generators, we have

$$\begin{aligned} \Delta^{(N)}(X_\pm) &= X_\pm \overbrace{e^{\frac{z}{2}X_3} \otimes e^{\frac{z}{2}X_3} \otimes \dots \otimes e^{\frac{z}{2}X_3}}^{(N-1)} + e^{-\frac{z}{2}X_3} \otimes X_\pm \otimes \overbrace{e^{\frac{z}{2}X_3} \otimes \dots \otimes e^{\frac{z}{2}X_3}}^{(N-2)} + \dots \\ &\quad + \overbrace{e^{-\frac{z}{2}X_3} \otimes e^{-\frac{z}{2}X_3} \otimes \dots \otimes e^{-\frac{z}{2}X_3}}^{(N-1)} \otimes X_\pm. \end{aligned} \tag{36}$$

Equivalently, this expression can be written from (5) as

$$\Delta^{(N)}(X_{\pm}) = \Delta^{(m)}(X_{\pm}) \otimes e^{\frac{z}{2}\Delta^{(N-m)}(X_3)} + e^{-\frac{z}{2}\Delta^{(m)}(X_3)} \otimes \Delta^{(N-m)}(X_{\pm}) \quad (37)$$

where $m = 1, \dots, (N - 1)$. By writing such N th deformed coproducts as $X_i^{(N)} := \Delta^{(N)}(X_i)$, the q -deformed CG system is defined by the Hamiltonian

$$H_z^{(N)} := C_z^{(N)} = \frac{1}{4} \left\{ \frac{\sinh\left(\frac{z}{2}X_3^{(N)}\right)}{z/2} \right\}^2 + X_+^{(N)}X_-^{(N)}. \quad (38)$$

The integrals of motion for $H^{(N)}$ are again $X_i^{(N)}$ ($i = 3, \pm$) and the sets $C_z^{(m)}$ and $C_z^{(N-m)}$ with $m = 1, \dots, (N - 1)$.

3.2.1. Evolution equations for the cluster variables. Now we can rewrite the N th coproduct (37) in terms of the cluster variables $X_i^{(m)}$

$$\begin{aligned} X_3^{(N)} &= X_3^{(m)} + X_3^{(N-m)} \\ X_{\pm}^{(N)} &= e^{-\frac{z}{2}X_3^{(m)}} X_{\pm}^{(N-m)} + X_{\pm}^{(m)} e^{\frac{z}{2}X_3^{(N-m)}}. \end{aligned} \quad (39)$$

However, it turns out to be convenient to introduce a new basis for the $sl_q(2)$ algebra, namely

$$S_3 = X_3 \quad S_{\pm} = e^{-zX_3/2} X_{\pm}. \quad (40)$$

In this new basis the Poisson brackets read

$$\{S_3, S_{\pm}\} = \pm 2S_{\pm} \quad \{S_+, S_-\} = \frac{\sinh(zS_3)}{z} e^{-zS_3} + 2zS_+S_- \quad (41)$$

the q -deformed coproduct is

$$\begin{aligned} \Delta(S_3) &= S_3 \otimes id + id \otimes S_3 \\ \Delta(S_{\pm}) &= e^{-zS_3} \otimes S_{\pm} + S_{\pm} \otimes id \end{aligned} \quad (42)$$

and the deformed Casimir reads

$$C_z = \frac{1}{4} \left\{ \frac{\sinh\left(\frac{z}{2}S_3\right)}{z/2} \right\}^2 + S_+S_-e^{zS_3}. \quad (43)$$

We remark that in the quantum mechanical case, the basis (40) is suitable in order to compute analytically the spectrum of some $sl_q(2)$ operators (see, for instance, [16–17]).

In terms of the N th deformed coproducts

$$S_i^{(N)} := \Delta^{(N)}(S_i) \quad (44)$$

the q -analogue (38) of the CG system is defined by the Hamiltonian

$$H_z^{(N)} := C_z^{(N)} = \frac{1}{4} \left\{ \frac{\sinh\left(\frac{z}{2}S_3^{(N)}\right)}{z/2} \right\}^2 + S_+^{(N)}S_-^{(N)} e^{zS_3^{(N)}}. \quad (45)$$

The integrals of motion for $H^{(N)}$ are again $S_i^{(N)}$ ($i = 3, \pm$) together with $C_z^{(m)}$ and $C_z^{(N-m)}$ ($m = 1, \dots, N - 1$). The new expressions for the N th coproduct in terms of the cluster variables $S_i^{(m)}$ are

$$S_3^{(N)} = S_3^{(m)} + S_3^{(N-m)} \quad S_{\pm}^{(N)} = e^{-zS_3^{(m)}} S_{\pm}^{(N-m)} + S_{\pm}^{(m)}. \quad (46)$$

The set of $3(N - 1)$ coupled nonlinear evolution equations

$$\dot{S}_i^{(m)} = \{S_i^{(m)}, H^{(N)}\} \quad m = 1, \dots, (N - 1) \quad i = 3, \pm \quad (47)$$

can be splitted into $(N - 1)$ copies of the following set of three equations:

$$\begin{aligned}\dot{S}_3^{(m)} &= 2(S_+^{(m)}\delta_- - S_-^{(m)}\delta_+)e^{z\delta_3} \\ \dot{S}_+^{(m)} &= 2z\delta_-e^{z\delta_3}S_+^{(m)}(\delta_+ - S_+^{(m)}) - \frac{\sinh(z\delta_3)}{z}S_+^{(m)} + \delta_+e^{z\delta_3}\frac{1 - e^{-2zS_3^{(m)}}}{z} \\ \dot{S}_-^{(m)} &= -2z\delta_+e^{z\delta_3}S_-^{(m)}(\delta_- - S_-^{(m)}) + \frac{\sinh(z\delta_3)}{z}S_-^{(m)} - \delta_-e^{z\delta_3}\frac{1 - e^{-2zS_3^{(m)}}}{z}\end{aligned}\quad (48)$$

where in order to eliminate the $S_i^{(N-m)}$ variables, we have used the $\delta_i \equiv S_i^{(N)}$ constants of motion given by

$$\delta_3 = S_3^{(m)} + S_3^{(N-m)} \quad \delta_{\pm} = e^{-zS_3^{(m)}}S_{\pm}^{(N-m)} + S_{\pm}^{(m)}.\quad (49)$$

Note that the limit $z \rightarrow 0$ of equations (48) reproduces the CG dynamics (29) and that we have again a constraint between the cluster variables coming from the constant of motion given by the m th deformed Casimir

$$C_z^{(m)} = \frac{1}{4} \left\{ \frac{\sinh\left(\frac{z}{2}S_3^{(m)}\right)}{z/2} \right\}^2 + S_+^{(m)}S_-^{(m)}e^{zS_3^{(m)}}.\quad (50)$$

Thus, once again the collective variables given by the coproduct give rise to a ‘separable’ deformed motion for each cluster. We also stress that such cluster dynamics is a typical mean-field dynamics: the evolution equations are the same for any m and the effect of the remaining degrees of freedom is hidden in the constants δ_i given by the N th coproducts of the generators. In a certain sense, each m th cluster moves within the mean-field generated by the others. This feature can be interpreted as the q -analogue of the fact that, at a quantum mechanical level, the mean-field approximation for the Gaudin Hamiltonian is exact.

3.2.2. Explicit solution. In order to solve the cluster equations (48), let us perform the following change of variables:

$$\tilde{S}_{\pm} = S_{\pm}^{(m)}\delta_{\mp}\quad (51)$$

$$\tilde{S}_3 = \exp(-2zS_3^{(m)}).\quad (52)$$

We obtain the new equations:

$$\dot{\tilde{S}}_{\pm} = \mp a(\tilde{S}_{\pm})^2 \pm b\tilde{S}_{\pm} \pm c(1 - \tilde{S}_3)\quad (53)$$

$$\dot{\tilde{S}}_3 = -2a\tilde{S}_3(\tilde{S}_+ - \tilde{S}_-)\quad (54)$$

where we have introduced the following constants:

$$a = 2z \exp(z\delta_3)\quad (55)$$

$$b = 2z\delta_- \delta_+ \exp(z\delta_3) - \frac{\sinh(z\delta_3)}{z}\quad (56)$$

$$c = \frac{\delta_+ \delta_- e^{z\delta_3}}{z}.\quad (57)$$

Summing and subtracting the above equations we get, respectively,

$$\dot{\tilde{S}}_+ + \dot{\tilde{S}}_- = -a(\tilde{S}_+ - \tilde{S}_-)\left(\tilde{S}_+ + \tilde{S}_- - \frac{b}{a}\right)\quad (58)$$

$$\dot{\tilde{S}}_+ - \dot{\tilde{S}}_- = -a(\tilde{S}_+^2 + \tilde{S}_-^2) + b(\tilde{S}_+ + \tilde{S}_-) + 2c(1 - \tilde{S}_3).\quad (59)$$

Thus we have

$$\frac{\dot{\tilde{S}}_3}{\tilde{S}_3} = -2a(\tilde{S}_+ - \tilde{S}_-) = 2\frac{\dot{\tilde{S}}_+ + \dot{\tilde{S}}_-}{\tilde{S}_+ + \tilde{S}_- - \frac{b}{a}} \tag{60}$$

which implies

$$\tilde{S}_3 = k_m \left(\tilde{S}_+ + \tilde{S}_- - \frac{b}{a} \right)^2 \tag{61}$$

where the number k_m is the integration constant that can be expressed in terms of the values of the m th cluster Casimirs as follows:

$$k_m^{-1} = \left(C_z^{(m)} - e^{-z\delta_3} C_z^{(N-m)} + \frac{1 - e^{-z\delta_3}}{2z^2} \right)^2. \tag{62}$$

Now it is natural to introduce

$$Y_+ = \tilde{S}_+ + \tilde{S}_- - \frac{b}{a} \quad Y_- = \tilde{S}_+ - \tilde{S}_- \tag{63}$$

entailing

$$\dot{Y}_+ = -aY_-Y_+ \tag{64}$$

$$\dot{Y}_- = -\frac{a}{2} \left[\left(1 + 4\frac{ck_m}{a} \right) Y_+^2 + Y_-^2 \right] + \frac{b^2 + 4ac}{2a}. \tag{65}$$

Setting

$$Z_+ = \frac{a}{(b^2 + 4ac)} \left(1 + 4\frac{ck_m}{a} \right)^{1/2} Y_+ \tag{66}$$

$$Z_- = \frac{a}{(b^2 + 4ac)} Y_- \tag{67}$$

we obtain the evolution equations

$$\dot{Z}_+ = -\Omega Z_+ Z_- \tag{68}$$

$$\dot{Z}_- = -\frac{1}{2}\Omega \left(Z_+^2 + Z_-^2 - \frac{1}{\Omega} \right). \tag{69}$$

By putting $\Omega = (b^2 + 4ac)$, the following equations are obtained:

$$\dot{Z}_+ \pm \dot{Z}_- = \mp \frac{\Omega}{2} \left((Z_+ \pm Z_-)^2 - \frac{1}{\Omega} \right) \tag{70}$$

whose solution reads

$$Z_+ \pm Z_- = \pm \frac{1}{\sqrt{\Omega}} \tanh \left(\frac{\sqrt{\Omega}}{2} (t - t_{\pm}^{(m)}) \right). \tag{71}$$

Note that

$$\frac{\sqrt{\Omega}}{2} = \sqrt{C_z^{(N)} + z^2(C_z^{(N)})^2} \tag{72}$$

and the limit $z \rightarrow 0$ of this expression gives the non-deformed period $\sqrt{C^{(N)}}$. Two remarks are here in order: first, on a given energy surface belonging to the open subset of the phase space $0 > C_z^{(N)} > -z^{-2}$, all the orbits are periodic with the same period; second, the nature of the deformed dynamics turns out to depend explicitly on the value of z .

3.3. A Gaudin system on a q -Poincaré algebra

We now consider the Poisson analogue of the (non-coboundary) quantum deformation of the $(1+1)$ Poincaré algebra $\mathcal{P}(1, 1)$ in terms of ‘light-cone’ coordinates:

$$\{X_3, X_\pm\} = \pm 2X_\pm \quad \{X_+, X_-\} = 0. \quad (73)$$

The deformed coproduct is (see [2] with $X_3 = 2K$ and $X_\pm = H \pm P$)

$$\begin{aligned} \Delta(X_3) &= X_3 \otimes id + id \otimes X_3 \\ \Delta(X_\pm) &= e^{-\frac{z}{2}X_3} \otimes X_\pm + X_\pm \otimes e^{\frac{z}{2}X_3} \end{aligned} \quad (74)$$

that, in spite of the non-triviality of the deformation, is still compatible with the non-deformed brackets (73). This deformation is isomorphic to the one first introduced in [18]. The known Casimir function for $\mathcal{P}(1, 1)$ (and, consequently, for this deformation) is

$$C_z = X_+ X_-. \quad (75)$$

Again, we shall make use of the new basis

$$S_3 = X_3 \quad S_\pm = e^{-zX_3/2} X_\pm \quad (76)$$

for which the $\mathcal{P}(1, 1)$ Poisson brackets read

$$\{S_3, S_\pm\} = \pm 2S_\pm \quad \{S_+, S_-\} = 2zS_+S_- \quad (77)$$

and the q -deformed coproduct is (42). The deformed Casimir function is

$$C_z = S_+ S_- e^{zS_3}. \quad (78)$$

We will consider again the ‘ q -Gaudin’ Hamiltonian given by the N th coproduct of the deformed Casimir

$$H^{(N)} = S_+^{(N)} S_-^{(N)} e^{zS_3^{(N)}}. \quad (79)$$

The associated dynamics is given by (47), and by using the constants of motion provided by the N th coproducts

$$\delta_3 = S_3^{(m)} + S_3^{(N-m)} \quad \delta_\pm = \exp(-zS_3^{(m)}) S_\pm^{(N-m)} + S_\pm^{(m)} \quad (80)$$

the evolution equations are splitted into $(N-1)$ copies ($m = 1, \dots, (N-1)$) of the set:

$$\begin{aligned} \dot{S}_3^{(m)} &= 2(S_+^{(m)} \delta_- - S_-^{(m)} \delta_+) e^{z\delta_3} \\ \dot{S}_+^{(m)} &= 2z\delta_- e^{z\delta_3} S_+^{(m)} (\delta_+ - S_+^{(m)}) \\ \dot{S}_-^{(m)} &= -2z\delta_+ e^{z\delta_3} S_-^{(m)} (\delta_- - S_-^{(m)}). \end{aligned} \quad (81)$$

In the limit $z \rightarrow 0$ these equations reproduce

$$\dot{X}_3^{(m)} = 2(X_+^{(m)} \delta_- - X_-^{(m)} \delta_+) \quad \dot{X}_+^{(m)} = \dot{X}_-^{(m)} = 0 \quad (82)$$

which leads to the (trivial) dynamics for the non-deformed Poincaré–Gaudin system.

3.3.1. Solutions for the cluster equations. The system (81) can be solved on the very same footing as the qCG system. The variables (52) now obey the simpler equations:

$$\dot{\tilde{S}}_\pm = \mp a(\tilde{S}_\pm)^2 \pm b\tilde{S}_\pm \quad (83)$$

$$\dot{\tilde{S}}_3 = -2a\tilde{S}_3(\tilde{S}_+ - \tilde{S}_-) \quad (84)$$

where

$$a = 2z \exp(z\delta_3) \quad (85)$$

$$b = 2z\delta_- \delta_+ \exp(z\delta_3) = \delta_- \delta_+ a = 2zC_z^{(N)}. \quad (86)$$

The evolution equation for \tilde{S}_\pm can be immediately integrated, yielding the ‘kink-shape’ solution:

$$\tilde{S}_\pm = \frac{b}{2a} \left[1 + \tanh \left(\mp \frac{b}{2} t - \phi_\pm^{(m)} \right) \right]. \tag{87}$$

Consequently, S_3 has the following time behaviour:

$$S_3^{(m)} = -(z^{-1} \log[(\cosh(bt + \phi_+ - \phi_-) + \cosh(\phi_+ + \phi_-))/2 - \cosh(\phi_+) \cosh(\phi_-)]) \tag{88}$$

which reduces to a linear function of t in the limit $z \rightarrow 0$. Note that one could add again a constant (depending on m) to the solution; such a constant has to vanish in the limit $z \rightarrow 0$. Recall that the constants of motion given by the cluster Casimirs

$$C_z^{(m)} = S_+^{(m)} S_-^{(m)} e^{z S_3^{(m)}} \quad \text{and} \quad C_z^{(N-m)} = S_+^{(N-m)} S_-^{(N-m)} e^{z S_3^{(N-m)}} \tag{89}$$

will also be related to the remaining integration constants.

3.4. A q -Ruijsenaars–Schneider system

We consider again the $P_z(1+1)$ algebra and the following Hamiltonian which under a specific symplectic realization [2] resembles the Ruijsenaars–Schneider system [19]:

$$H^{(N)} = X_+^{(N)} + X_-^{(N)} = (S_+^{(N)} + S_-^{(N)}) e^{z S_3^{(N)}/2}. \tag{90}$$

This Hamiltonian Poisson commutes with the N Casimirs

$$C^{(m)} = S_+^{(m)} S_-^{(m)} e^{z S_3^{(m)}} \quad m = 1, \dots, N. \tag{91}$$

On the other hand, since $\{X_+^{(N)}, X_-^{(N)}\} = 0$, we have two additional constants of motion (the N th Casimir is just the product of them) given by

$$\delta_\pm = X_\pm^{(N)} = e^{z S_3^{(N)}/2} S_\pm^{(N)} = e^{-z(S_3^{(m)} - S_3^{(N-m)})/2} S_\pm^{(N-m)} + S_\pm^{(m)} e^{z(S_3^{(m)} + S_3^{(N-m)})/2}. \tag{92}$$

Moreover, note that $\delta_+ + \delta_- = H^{(N)}$. By using such constants we get the following sets of separated equations for the cluster variables $S_i^{(m)}$:

$$\begin{aligned} \dot{S}_3^{(m)} &= 2(S_+^{(m)} - S_-^{(m)}) e^{z S_3^{(N)}/2} \\ \dot{S}_+^{(m)} &= z S_+^{(m)} \left((\delta_+ + \delta_-) - 2 S_+^{(m)} e^{z S_3^{(N)}/2} \right) \\ \dot{S}_-^{(m)} &= -z S_-^{(m)} \left((\delta_+ + \delta_-) - 2 S_-^{(m)} e^{z S_3^{(N)}/2} \right). \end{aligned} \tag{93}$$

Since it is immediate to check that

$$\dot{S}_3^{(N)} = 2(\delta_+ - \delta_-) \longrightarrow S_3^{(N)} = 2(\delta_+ - \delta_-)t + \beta \tag{94}$$

through the change of variables

$$Y_\pm^{(m)} = S_\pm^{(m)} e^{z(\delta_+ - \delta_-)t + \beta} \tag{95}$$

we get the explicit solution for the variables $Y_\pm^{(m)}$:

$$Y_\pm^{(m)} = \frac{\delta_\pm}{2} \left\{ 1 + \tanh \left(\pm z \delta_\pm (t - t_\pm^m) \right) \right\} \tag{96}$$

which is again ‘kink-shape’. Note that, when compared with (87), the exponential term (95) for the $S_\pm^{(m)}$ variables and the linear motion for $S_3^{(N)}$ (94) are the only dynamical differences between both systems, both of them due to the fact that the Hamiltonian (93) does not commute with $S_3^{(N)}$.

4. Poisson–Lie dynamics on a solvable group

The definition of all the cluster variables for the previous systems is always based on the same type of deformed coproduct (namely (46) and its S -basis form (42)), although each particular dynamics is specialized by a given Poisson structure and Hamiltonian. As we shall see in what follows, this observation can be rephrased in group theoretical terms by considering such coproduct as the group law for a given solvable Lie group. Namely, let us consider the following element g_z of a three-dimensional Lie group G_z :

$$g_z = \begin{pmatrix} e^{-zA} & 0 & C \\ 0 & e^{-zA} & B \\ 0 & 0 & 1 \end{pmatrix}. \quad (97)$$

The group parameters are A , B and C (z is a real constant). It is straightforward to compute the Lie algebra generators associated with A , B and C which are, respectively,

$$D = \begin{pmatrix} -z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (98)$$

These generators close the following three-dimensional solvable Lie algebra:

$$[D, P_1] = -zP_1 \quad [D, P_2] = -zP_2 \quad [P_1, P_2] = 0 \quad (99)$$

that contains a dilation D together with two Euclidean translations P_1 and P_2 . Obviously, when $z \neq 0$ such a parameter can be reabsorbed through the automorphism $D \rightarrow D/z$. On the other hand, the $z \rightarrow 0$ limit of G_z is the three-dimensional Abelian group (see [20–22] for the general connection between quantum algebras and dual Poisson–Lie groups).

A natural coalgebra structure on $\text{Fun}(G_z)$ is defined through the coproduct given by the matrix multiplication on a representation on the group

$$\Delta(g_z) := g_z \otimes g_z. \quad (100)$$

From the representation (97) such a coproduct takes the following values on the group entries:

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 \\ \Delta(e^{-zA}) &= e^{-zA} \otimes e^{-zA} \\ \Delta(B) &= e^{-zA} \otimes B + B \otimes 1 \\ \Delta(C) &= e^{-zA} \otimes C + C \otimes 1. \end{aligned} \quad (101)$$

Note that the second expression is equivalent to the fact that

$$\Delta(A) = 1 \otimes A + A \otimes 1. \quad (102)$$

Therefore, under the identification

$$A \equiv S_3 \quad B \equiv S_+ \quad C \equiv S_- \quad (103)$$

the coproduct (42) is just (101). Therefore, all the N th coalgebra systems described above are defined on N tensor copies of different Poisson algebras of (smooth) functions on G_z . In general, any Poisson structure on $\text{Fun}(G_z)$ for which the group multiplication (101) is a Poisson map endows $\text{Fun}(G_z)$ with the structure of a Poisson–Lie group (see, for instance, [23, 24] and references therein). Hence, all the previous results can be interpreted as examples of PL dynamics on $\text{Fun}(G_z)$.

4.1. The PL $sl_q(2)$ dynamics

Through (103), the $sl_q(2)$ Poisson coalgebra (41) can be identified as a Poisson–Lie structure on $\text{Fun}(G_z)$ with the following Poisson brackets

$$\{A, B\} = 2B \quad \{A, C\} = -2C \quad \{B, C\} = \frac{1 - e^{-2zA}}{z} + 2zBC \quad (104)$$

coproduct (101) and Casimir function (43)

$$I_z = \frac{1}{4} \left(\frac{\sinh zA/2}{z/2} \right)^2 + BC e^{zA}. \quad (105)$$

From this perspective, the q -CG system is just obtained when we consider the evolution under the particular Hamiltonian $H^{(N)} = \Delta^{(N)}(I_z)$. Note that the non-deformed CG system is obtained in the limit $z \rightarrow 0$, which corresponds to the three-dimensional Abelian group G_0 where the additive group law for the parameters is just the primitive coproduct (22).

4.2. The PL q -Euclidean dynamics

In the same way, the Poisson $P_z(1+1)$ algebra (77) can be thought of as a different Poisson–Lie structure on $\text{Fun}(G_z)$ endowed with the Poisson brackets:

$$\{A, B\} = 2B \quad \{A, C\} = -2C \quad \{B, C\} = 2zBC. \quad (106)$$

The Casimir function is the analogue of (78):

$$I_z = BC e^{zA}. \quad (107)$$

Now, the q -Gaudin system on the Poincaré algebra is defined by the Hamiltonian

$$H^{(N)} = \Delta^{(N)}(BC e^{zA}) \quad (108)$$

whilst the Hamiltonian for the q -Ruijsenaars–Schneider system is

$$H^{(N)} = \Delta^{(N)}((B + C) e^{zA/2}). \quad (109)$$

We stress that both systems are defined with respect to the same PL bracket (106).

4.3. PL $sl_q^\kappa(2)$ dynamics and contractions

From this PL point of view it would be natural to explore the dynamics induced from other Poisson–Lie structures on G_z and to exploit the PL structure in order to extract more dynamical information. For instance, we could realize that (104) and (106) can be simultaneously written as a one-parameter family of PL brackets:

$$\{A, B\} = 2B \quad \{A, C\} = -2C \quad \{B, C\} = \kappa \frac{1 - e^{-2zA}}{z} + 2zBC \quad (110)$$

where the parameter $\kappa \geq 0$ can be interpreted as a contraction parameter: if $\kappa = 1$ we have the $sl_q(2)$ structure and the limit $\kappa \rightarrow 0$ leads to the q -Poincaré one. Note that (110) is compatible with the group coproduct (101) for any value of κ . The Casimir function for this more general bracket is

$$I_z^\kappa = \frac{\kappa}{4} \left(\frac{\sinh zA/2}{z/2} \right)^2 + BC e^{zA}. \quad (111)$$

It is easy to check that equations for the cluster variables (81) of the q -Gaudin–Poincaré system are just the $\kappa \rightarrow 0$ contraction of the q -CG equations (48), provided the latter system is defined by using the Poisson algebra $sl_q^\kappa(2)$ (110) instead of the $sl_q(2)$ one (104).

Finally we remark that, since the Poisson structure (110) is quadratic in terms of the entries of g_z (note that the number 1 is one of such entries), it is possible to rewrite (110) in the form

$$\{G_z \otimes G_z\} = [r, G_z \otimes G_z] \quad (112)$$

where r is a 9×9 constant classical r -matrix. We also recall that, in general, a suitable quantization of Poisson–Lie groups gives rise to quantum groups, and some quantum versions of different PL brackets on the group G_z were already given in [20].

Acknowledgments

AB has been partially supported by MCyT and Junta de Castilla y León (España, Projects BFM2000-1055 and BU04/03). OR has been partially supported by INFN and by MIUR (COFIN2001 ‘Geometry and Integrability’).

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